Stable stationary and quasiperiodic discrete vortex breathers with topological charge $S=2$

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We demonstrate the stability of a stationary vortex breather with vorticity $S=2$ in the two-dimensional discrete nonlinear Schrödinger model for a square lattice and also discuss the effects of exciting internal sites in a vortex ring. We also point out the fundamental difficulties of observing these solutions with current experimental techniques. Instead, we argue that relevant initial conditions will lead to the formation of quasiperiodic vortex breathers.

Over the last decades much interest has been paid to the concept of self-localization of solutions to nonlinear lattice equations [1,2]. Generally employed as models of periodic nonlinear systems that are abundant in nature, the equations incorporate nonlinearity and coupling, or tunneling, between adjacent sites of the lattice, two effects which, when in balance, lead to localization. The existence of intrinsic localized modes (also known as discrete breathers or lattice solitons) has been proven rigorously under very general conditions [3,4], and experimental observations are numerous [5–14]. Recent experimental activity has to an increasing extent focused on two-dimensional systems—e.g., optically induced nonlinear photonic lattices [15]. In contrast to excitations in one spatial dimension, higher-dimensional localized modes can carry angular momentum, which due to the discrete symmetry is generally not a conserved quantity, manifested as a screw dislocation of the phase on a closed contour encircling the excitation. In such a *discrete vortex soliton* the angular momentum is proportional to the topological charge S, or vorticity—i.e., the number of complete 2π twists of the phase on the contour. In simulations of continuous models of nonlinear periodic systems localized vortices have been found, e.g., for a Bose-Einstein condensate trapped in an optical potential [16,17] and optically induced waveguides in a Kerr medium (cubic nonlinearity) [18] or a photorefractive crystal (saturable nonlinearity) [19]. Also asymmetric vortices have been suggested [20].

A fruitful, and quite simple, approach to nonlinear periodic systems is the use of Wannier function expansion to reduce the system to a nonlinear lattice equation [21]. In the tight-binding approximation of nearest-neighbor coupling the generic equation that incorporates nonlinearity and coupling at lowest order is the discrete nonlinear Schrödinger (DNLS) equation, which in two dimensions for a square lattice reads

$$i\dot{\Psi}_{n,m} = C(\Psi_{n-1,m} + \Psi_{n+1,m} + \Psi_{n,m-1} + \Psi_{n,m+1}) + |\Psi_{n,m}|^2\Psi_{n,m},$$
$$(1)$$

where $\Psi_{n,m}$ is a complex field quantity, the overdot represents differentiation with respect to time, and the constant C

determines the strength of the intersite coupling. The relevance of the DNLS model for physical systems is noticeable in the large amount of research devoted to the equation (see [22] for a review) and also in some phenomenological predictions that have been experimentally verified in one-dimensional systems [5,23,24]. In the context of Eq. (1), discrete vortex solitons were computed in [25–27]. Of the fundamental vortices discovered only the $S=1$ and $S=3$ vortices were found to be stable. The rigorous analysis made in [28] of the persistence and stability of DNLS vortices with the main excited sites orientated along a square contour in the lattice substantiates these results. It has also been suggested that the unstable $S=2$ vortex can be stabilized by an impurity [29].

The restrictions set on the vorticity by the discrete lattice symmetry are discussed in [30–32]. However, the vorticity in these papers, defined as the phase twist belonging solely to the plane-wave part of the angular Bloch mode connected to the solution, is not necessarily equal to the overall vorticity as defined here and in [26–28]. Therefore, the statement in [30–32] that a lattice with a discrete point symmetry of order n (the symmetry group is $\mathcal{C}_n$ or $\mathcal{C}_{nv}$ and $n=4$ for a square lattice) cannot support vortices of charge larger than $n/2$ is not a contradiction to the results in [27,28] and in the present paper, but merely due to a lack of preciseness in definition. The discrepancy is that the periodic part of the angular Bloch mode may contribute any $2\pi n$ multiple to the total phase twist, where n is the symmetry order, in addition to the contribution from the plane-wave part, without failing to fulfill the symmetry condition set by the discrete point group. The high-order ($>n/2$) vortices can be related to the vorticity of the Bloch modes by a process somewhat analogous to a reduction from a periodic to a reduced zone scheme in solid-state physics. In particular, a $S=3$ vortex in a square lattice can be reduced to a Bloch mode consisting of a plane-wave part with a phase twist of -2π and a periodic part with a phase twist of $+8\pi$—i.e., what would be referred to as a vortex with charge -1 in [30–32]. Moreover, the statement (not proven) in the same papers that there are no true vortices with charge $n/2$ (n even) is incorrect as all possible solutions are not taken into account. Examples can be found below and in [26–28].

Experimentally, stable $S=1$ vortices have been observed

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in photonic lattices [33,34], whereas there is, to the best of our knowledge, yet no observation of higher-order vortices in periodic nonlinear systems. Recently the observation was reported of necklacelike solitons, with actual charge $S=0$, being formed from an initial condition with high vorticity [35].

In this paper we study numerically the existence and stability of discrete vortex solitons of the DNLS equation (1) different from the most fundamental previously obtained [26,27]. We show that with respect to stability the orientation of the excitations in the lattice is of importance, as well as the effect of exciting internal sites in a vortex ring. Particularly we report a stable $S=2$ stationary vortex, previously not thought to exist, and show its persistence in dynamical simulations. We stress that our stable $S=2$ vortex has a different structure than the unstable $S=2$ solutions previously studied. Moreover, we argue that trying to excite the vortex from an experimentally relevant initial condition will lead to the creation of a stable quasiperiodic vortex breather.

Obtaining numerically exact solutions to Eq. (1) is much simplified by its global phase invariance, since this permits a gauge transformation of monochromatic time-periodic solutions to a frame of reference where the solutions are stationary. Thus, taking $\Psi_{n,m}(t)=\psi_{n,m}e^{-i\Lambda t}$ reduces Eq. (1) to

$$-\Lambda\psi_{n,m} + C(\psi_{n-1,m}+\psi_{n+1,m}+\psi_{n,m-1}+\psi_{n,m+1})$$
$$+ |\psi_{n,m}|^2\psi_{n,m} = 0. \tag{2}$$

In the anticontinuous limit ($C=0$) of Eq. (2) the system is equivalent to a set of uncoupled anharmonic oscillators and solutions are trivial. For each site we may take $\psi_{n,m} \in \{0, \sqrt{\Lambda}e^{i\alpha}\}$ with $\alpha \in]-\pi, \pi]$, thus creating an arbitrary configuration of excited and resting sites. However, the choices of the relative phases are not arbitrary if the solution is to persist up to some finite coupling [28,36]. A consequence of the phase invariance of Eq. (1) is that the norm, or excitation number, $\mathcal{N}=\Sigma_{n,m}\mathcal{N}_{n,m}=\Sigma_{n,m}|\Psi_{n,m}|^2$ is a conserved quantity. This can be expressed in the form of a discrete continuity equation

$$\dot{\mathcal{N}}_{n,m} + \mathcal{J}_{n,m}^{(x)} - \mathcal{J}_{n-1,m}^{(x)} + \mathcal{J}_{n,m}^{(y)} - \mathcal{J}_{n,m-1}^{(y)} = 0, \tag{3}$$

where the norm current density is

$$\mathcal{J}_{n,m}^{(x)} = -2C\,\mathrm{Im}\{\Psi_{n,m}^*\Psi_{n+1,m}\}$$
$$= 2C\sqrt{\mathcal{N}_{n,m}\mathcal{N}_{n+1,m}}\sin(\theta_{n+1,m}-\theta_{n,m}), \tag{4}$$

if $\Psi_{n,m}=\sqrt{\mathcal{N}_{n,m}}e^{-i\theta_{n,m}}$, and where $\mathcal{J}_{n,m}^{(y)}$ is given by Eq. (4) with the roles of n and m interchanged. Thus, for a stationary solution, implying a constant intensity ($\dot{\mathcal{N}}_{n,m}=0$), Eqs. (3) and (4) impose a set of conditions on the relative phases of the solution. Only trivial solutions subject to this constraint may be numerically continued to nonzero coupling by an iterative Newton method [37,38]. Further, the linear stability of the solutions can be examined by considering the evolution of a small perturbation to the solution. Inserting $[\psi_{n,m}+\epsilon_{n,m}(t)]e^{-i\Lambda t}$ into Eq. (1) and keeping only first-order terms in $\epsilon_{n,m}$ leads to a linear eigenvalue problem for the growth rates of $\epsilon_{n,m}$. Since the problem is infinitesimally

symplectic, $\pm\lambda$ and $\pm\lambda^*$ will be simultaneous eigenvalues and linear stability follows if all eigenvalues have zero real part [39].

The most fundamental vortex mode with $S=1$ is obtained by exciting four sites in a square configuration, with a relative phase difference of $\pi/2$ between neighboring sites [26,40]. If, without loss of generality, the frequency is fixed at $\Lambda=1$, this solution will be stable for $C<0.1512$, while the square configuration placed diagonally in the lattice with a zero amplitude mediating site in the center will be stable for $C<0.1525$ (see Table I). The calculations have been made on a lattice, typically of size 21×21, but have been checked for other sizes, with periodic boundary conditions. Also for the higher-order vortices the diagonally aligned configurations have a larger interval of stability than the configurations aligned along the lattice, which are of the type classified and investigated in [28]. This may be expected because of the larger distance between the excited sites, which means that a stronger coupling is needed before any destabilizing interactions come into play. In the large coupling limit the system approaches the continuum two-dimensional nonlinear Schrödinger equation where all solitons are unstable and it is therefore expected that instabilities set in at increasing coupling. Of the initially linearly stable configurations presented in Table I, the transition to unstable solutions is always the result of a pair of eigenvalues colliding with the band of extended eigenmodes creating four complex eigenvalues off the imaginary axis.

We especially note the *stability of the $S=2$ vortex* for $C<0.1348$ (mode i in Table I), which as far as we know has not been reported for any related systems. In [29] it was shown how the unstable $S=2$ vortex (mode h in Table I) can be stabilized by making the center site inert, but we would like to stress that the stability of the vortex presented here is not due to any impurities in the system. Further, though larger in diameter the vortex still only incorporates eight main excited sites as can be seen in Fig. 1. The stability has also been checked in dynamical simulations where the exact solution is used as an initial condition against a uniformly distributed background noise. The simulations concur with the results from the linear stability analysis. But even when the solution is linearly unstable it can survive for tens of internal breather oscillations before the instability develops in simulations. However, this time will decrease as the coupling increases and as a second and third quartet of eigenvalues with nonzero real part emerge at $C=0.1563$ and $C=0.1583$, respectively. A further increase of the number of excited sites for $S=2$ vortices seems to lead to unstable configurations. Vortices on a square contour with 12 and 16 excited sites and 4 and 9 internal unexcited sites, respectively, are unstable as shown in [28].

With an increasing radius of the vortex rings in the lattice it is also interesting to investigate the effects of exciting the internal sites, within the constraints set by Eq. (3) as discussed above. It is, e.g., possible to stabilize the large $S=1$ vortex (mode d in Table I) by exciting the center site (mode e). A similar configuration for the higher-order vortex rings will only give unstable excitations when continued beyond the anticontinuous limit. In fact, no stable modes with internal sites excited have been found for $S=2$ and $S=3$, which is

TABLE I. Vorticity and phase configuration at the anticontinuous limit $C=0$ of some stationary vortex breathers, where $\cdot$ is used to denote unexcited sites. The interval of stability obtained from continuation of the initial configurations is also shown.

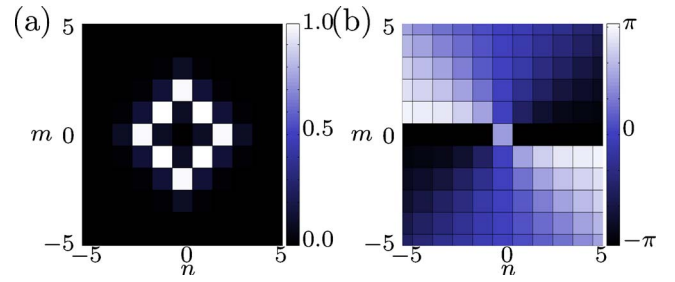|	S			Phase configuration (α)			Stable
a	1		0		$-\pi/2$		$C<0.1512$
			$\pi/2$		π		
b	1			0			$C<0.1525$
			$\pi/2$	$\cdot$	$-\pi/2$		
				π			
c	1		$\pi/4$	0	$-\pi/4$		No
			$\pi/2$	$\cdot$	$-\pi/2$		
			$3\pi/4$	π	$-3\pi/4$		
d	1			0			No
			$\pi/4$	$\cdot$	$-\pi/4$		
		$\pi/2$	$\cdot$	$\cdot$	$\cdot$	$-\pi/2$	
			$3\pi/4$	$\cdot$	$-3\pi/4$		
				π			
e	1			0			$C<0.1177$
			$\pi/4$	$\cdot$	$-\pi/4$		
		$\pi/2$	$\cdot$	0	$\cdot$	$-\pi/2$	
			$3\pi/4$	$\cdot$	$-3\pi/4$		
				π			
f	1			0			No
			$\pi/4$	0	$-\pi/4$		
		$\pi/2$	$\pi/2$	$\cdot$	$-\pi/2$	$-\pi/2$	
			$3\pi/4$	π	$-3\pi/4$		
				π			
g	1			0			$C<0.0610$
			$\pi/4$	π	$-\pi/4$		
		$\pi/2$	$-\pi/2$	$\cdot$	$\pi/2$	$-\pi/2$	
			$3\pi/4$	0	$-3\pi/4$		
				π			
h	2		$\pi/2$	0	$-\pi/2$		No
			π	$\cdot$	π		
			$-\pi/2$	0	$\pi/2$		
i	2			0			$C<0.1348$
			$\pi/2$	$\cdot$	$-\pi/2$		
		π	$\cdot$	$\cdot$	$\cdot$	π	
			$-\pi/2$	$\cdot$	$\pi/2$		
				0			
j	3		$3\pi/4$	0	$-3\pi/4$		$C<0.0704$
			$-\pi/2$	$\cdot$	$\pi/2$		
			$\pi/4$	π	$-\pi/4$		
k	3			0			$C<0.1266$
			$3\pi/4$	$\cdot$	$-3\pi/4$		
		$-\pi/2$	$\cdot$	$\cdot$	$\cdot$	$\pi/2$	
			$\pi/4$	$\cdot$	$-\pi/4$		
				π			

FIG. 1. (Color online) (a) $|\Psi_{n,m}|$ and (b) $\arg\{\Psi_{n,m}\}$ for a stable stationary vortex of topological charge $S=2$ for coupling $C=0.1$ (mode i in Table I).

in contrast to the modes e and g for $S=1$. The stability of the latter for a focusing nonlinearity, as compared to the mode f, can be understood from the fact that it favors out-of-phase oscillations between neighboring sites [41].

For experimental systems where the model (1) is of relevance, like a regular bundle of optic fibers or other nonlinear waveguides or like photonic crystals with optically induced waveguides, it is in principle possible to excite single sites of the lattice by a focused laser beam. Also in the context of a Bose-Einstein condensate loaded into an optical lattice [42] a detailed control can be exercised by vacating unwanted sites through laser-heated evaporation of the condensate at these sites. Although the DNLS model is a very good description for the systems with a strong optical lattice potential, which is demonstrated by the concurrence between the phenomenology of the model and a number of experiments [5,23,24,33,34], the observation of some of the solutions presented here will present technical difficulties. This is, despite the possibility of controlling excitations on individual sites, mainly due to the difficulty in experiments to control the individual phase relations between the sites. Current experimental techniques rely on phase masks (vortex masks) to give a circularly symmetric light beam the desired vorticity before it is focused on the face of a waveguide array or a photonic crystal [33–35]. A relevant initial condition in these circumstances is (compare [33])

$$f(r,\theta) = Ar^{|S|}e^{iS\theta}e^{-w(r-r_0)^2}, \qquad (5)$$

where S is the vorticity and the circular coordinates are related to the lattice indices by $re^{i\theta}=n+im$ for an on-site excitation with unit lattice spacing. To get an off-site excitation a half integer shift is introduced on the lattice indices. The parameter A determines the amplitude, w is the inverse width of the beam ring, and r_0 is the radius of the ring, which in previous investigations has been set to $r_0=0$ to achieve the fundamental vortex modes. Indeed, the initial condition (5) with $S=1$, $A=2$, $w=1$, and $r_0=0$ (see Fig. 2) against a noisy background for $0.03\lesssim C\lesssim0.1$ will evolve into a charge one stationary vortex corresponding to mode b in Table I.

To obtain the stationary solutions of Eq. (1) it is important that the excited sites, not counting the decaying tail, have the same amplitude. Since the frequency of oscillation is related to the intensity of the main excited sites and directly determined by their value at the anticontinuous limit, a mix of
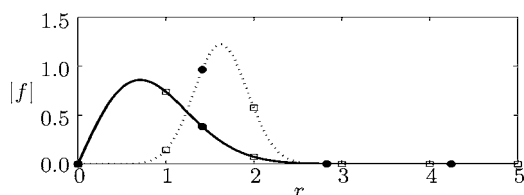
FIG. 2. The radial dependence of the initial condition (5) for two different values of the parameters: $S=1$, $A=2$, $w=1$, $r_0=0$ (solid line) and $S=2$, $A=0.5$, $w=5$, $r_0=1.5$ (dotted line). The squares ($\square$) indicate the values at sites along a lattice direction and the circles ($\bullet$) values along a direction diagonally in the lattice for unit lattice spacing.

amplitudes will lead to a mix of different frequencies. The generic case, and the only possibility giving a localized exact solution as explained in [25,43], is a solution with two incommensurate frequencies of oscillation forming a quasiperiodic excitation. Thus, a ringlike initial condition with
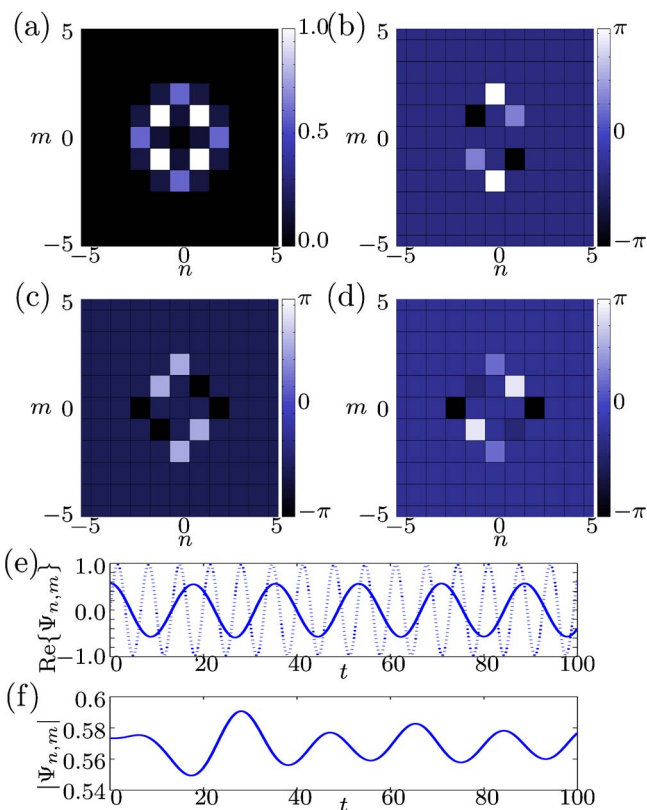
FIG. 3. (Color online) The dynamics of the initial condition with $S=2$ in Fig. 2 against a noisy background for $C=0.049$. (a) $|\Psi_{n,m}(t=0)|$ and (b) $\arg\{\Psi_{n,m}(t=0)\}$, where only the phase of the main excited sites are shown for clarity. As the vortex evolves, without much change to the initial amplitudes, the vorticity will change from $S=2$ in (b) to $S=0$ in (c) at $t=2.6$ and to $S=-2$ in (d) at $t=5.3$ and back again. The quasiperiodicity is also clearly seen in plot (e) of the real parts of the amplitudes at site $(-2,0)$ (solid line) and site $(-1,1)$ (dotted line), but as can be inferred from plot (f) of $|\Psi_{-2,0}|$ it is not an exact two-frequency quasiperiodic solution. The calculations were made on a 21×21 lattice with periodic boundary conditions.
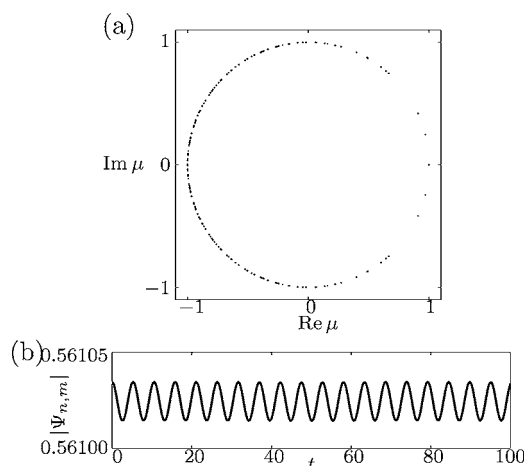

FIG. 4. (a) The distribution of Floquet multipliers for an exact quasiperiodic vortex breather with $\Lambda_0=0.3450$ and $\Lambda_b=0.5950$ for $C=0.049$ showing its linear stability. The system size is 11×11. (b) The time evolution of $|\Psi_{-2,0}|$ for the same solution illustrating the periodicity of the oscillations with period $T_b/2=\pi/\Lambda_b=5.2800$ as is expected for an exact solution. Compare this with Fig. 3(f) and note also that the magnitude of the oscillations differ by about three orders of magnitude.

larger radius, $r_0\neq0$ in Eq. (5), which, e.g., would be the relevant choice to obtain the stable $S=2$ stationary vortex discussed above, will generally lead to the excitation of a quasiperiodic vortex breather. This is illustrated in Fig. 2, where the resulting amplitudes along a lattice direction and diagonally in the lattice are marked. In principle, the initial condition can be chosen so that the main excited sites in the two directions get the same amplitude, but in practice this is not feasible. In simulations even a slight deviation of the amplitudes will lead to quasiperiodicity. The dynamics of the initial condition in Fig. 2 is illustrated in Fig. 3. The initial condition will generally not yield an exact two-frequency quasiperiodic vortex. This can be seen from Fig. 3(f) since the period of oscillation of $|\Psi_{n,m}|$ has no relation to the change of vorticity in Figs. 3(b)–3(d) with period $T\sim10.6$. However, the dynamics will in the general case stay very close to an exact solution, the existence of which can be inferred from its construction from the anticontinuous limit. As such we may regard the solution as constructed from two interlaced quadrupoles with different amplitudes—i.e., different incommensurate frequencies Λ_0 and Λ_1—at $C=0$. Due to the quasiperiodicity, the vortex will not have a well-defined vorticity, but will instead oscillate between $S=2$ and $S=-2$ with frequency $\Lambda_b=\Lambda_1-\Lambda_0$ which is associated with the time-dependent phase difference between the two quadrupoles. This will also be manifested as a periodically changing flow of the norm current density (4) in the vortex ring, as required from Eq. (3) for an intensity $\mathcal{N}_{n,m}$ which is periodic in time. This phenomenon of "charge flipping" was numerically observed in [20] for a related system when applying an amplitude perturbation to a stationary (time-periodic) vortex with $S=1$. In the model (1) we conclude that the effect is due to the existence of exact quasiperiodic vortex breathers.

To obtain numerically exact quasiperiodic vortex breathers the method with continuation from the anticontinuous limit, as described in [25,43], is used. Let the excited sites have magnitudes $\sqrt{\Lambda_0}$ and $\sqrt{\Lambda_1}$, with appropriate relations between the phases, for $C=0$. After shifting Eq. (1) to a frame of reference rotating with frequency Λ_0 the problem can be reduced to finding time-periodic solutions with frequency $\Lambda_b = \Lambda_1 - \Lambda_0$ for increasingly larger coupling. Numerically this is achieved by integrating the system over one period and searching for the fixed points of the corresponding map with a Newton method scheme. In addition, linear stability is simply investigated by standard Floquet analysis, as described in [25,43], leading to a symplectic eigenvalue problem for the Floquet multipliers; i.e., μ, μ^*, $1/\mu$, and $1/\mu^*$ will be simultaneous multipliers and linear stability follows if all multipliers are on the unit circle [39]. This will for each configuration of excited sites at $C=0$ give a two-parameter family of solutions. Taking $\Lambda_0 = 0.3450$ and $\Lambda_b = 0.5950$, with sites arranged in a ring and with appropriate phase relations, will lead to a set of solutions closely resembling the ones created from the initial condition with $S=2$ in Fig. 2. As seen in Fig. 4(a) this quasiperiodic vortex breather will be stable for $C=0.049$. Using the exact solution as an initial condition in Eq. (1) will produce a dynamics that is practically identical to the scenario in Figs. 3(a)–3(e). The only observable difference is the oscillations in $|\Psi_{n,m}|$ that now will be periodic with period $T_b/2 = \pi/\Lambda_b$, as illustrated in Fig. 4(b). From the linear stability analysis of the exact quasiperiodic solution it is revealed that it is essentially stable (except for windows of very weak instability associ-

ated with phonon resonances) up to $C=0.061$, where a complex instability sets in that will break up the solution. This is in very good agreement with the dynamical simulations of the initial condition in Fig. 2, where the vortex breather will break up and generally form a single quadrupole for $C>0.060$. With other choices of the frequencies a qualitatively different behavior of the solutions is possible. It is, e.g., possible to have a nondestructive phonon resonance, which theoretically occurs at $C=(\Lambda_b - \Lambda_0)/4$, before any destructive instabilities; i.e., there exist also stable quasiperiodic phonobreathers with a small nondecaying tail (cf. [43]). Further, we have also observed the creation of quasiperiodic solutions corresponding to a vorticity $S=1$. This can be achieved by, e.g., taking the initial condition (5) with $A=2$, $S=1$, $w=0.5$, and $r_0=0$.

In conclusion, we have demonstrated the existence and stability of both stationary and quasiperiodic vortex breathers with topological charge $S=2$. We further argue that current experimental techniques are more likely to lead to an observation of the latter than the former.

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